# Anomalous phase synchronization in two asymmetrically coupled oscillators in the presence of noise

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(Received 28 February 2005; published 23 December 2005)

We study the route to synchronization in two noisy, nonisochronous oscillators. Anomalous phase synchronization arises if both oscillators differ in their respective value of nonisochronicity and it is characterized by a strong detuning of the oscillator frequencies with the onset of coupling. Here we show that anomalous synchronization, both in limit-cycle or chaotic oscillators, can considerably be enlarged under the influence of asymmetrical coupling and noise. In these systems we describe a number of noise induced effects, such as an inversion of the natural frequency difference and coupling induced desynchronization of two identical oscillators. Our results can be explained in terms of a noisy particle in a tilted washboard potential.

DOI: 10.1103/PhysRevE.72.066216 PACS number(s): 05.45.Xt

## I. INTRODUCTION

Synchronization in interacting oscillators is one of the most fundamental problems in nonlinear dynamics [1–4]. In any real application the oscillators are necessarily nonidentical and vary in their system parameters. Synchronization then arises as an interplay between the interaction and the frequency mismatch of the two oscillators. Of special interest is the phenomenon of phase synchronization in which coupling can overcome the difference of natural frequencies and the oscillators are mutually entrained to a common locking frequency. Phase synchronization is an ubiquitous phenomenon and arises naturally in many areas of physics and natural systems. It has been observed in limit cycle and in chaotic oscillators and it appears in coupled pairs of oscillators, as well as in large ensembles of oscillators [2,5–9], and even in two interacting populations of oscillators [10]. Biological examples of synchronization include synchronous flashing fireflies [11], firing of neurons [12,13] and oscillating population numbers [9,14].

Usually the introduction of coupling simply leads to synchronization among the oscillators. However coupling may also give rise to different effects including oscillation death [8,15,16] and inhibition of synchronization [17,18]. Recently, a new route to phase synchronization has been described where small coupling firstly enlarges the natural frequency difference between the oscillators, whereas phase synchronization sets in only for larger values of coupling strength [19,20]. This "anomalous" phase synchronization has been demonstrated to arise naturally in a large class of oscillator types and coupling topologies [19,20], including two interacting spatially extended systems [21], and it has experimentally been confirmed in two coupled Chua's circuits [22].

In this paper, we investigate anomalous synchronization in two nonidentical oscillators under the influence of noise and asymmetrical coupling. In the previous investigations it was observed that anomalous synchronization is rather weak in the common case that the oscillators have similar values of nonisochronicity [19,20]. Here, we show that this restriction does not hold any more if the coupling symmetry is broken. In this case we find that anomalous synchronization is dominant even if the two oscillators have identical nonisochronicities.

Anomalous phase synchronization so far has only been described in deterministic systems. In contrast, the interplay between usual phase synchronization and stochasticity has intensively been studied [3]. Most prominently noise is able to induce phase slips and in this way can dephase otherwise locked oscillators. However, even in this situation phase synchronization can be defined in a statistical sense [3]. On the other hand, it was shown that correlated noise may enhance or promote the amount of synchrony between two oscillators [23–25]. Here we investigate the interplay between anomalous synchronization and noise. We find that in the presence of noise anomalous effects are much enlarged and we demonstrate a number of interesting noise induced phenomena in nonisochronous oscillators. In particular, in such systems the onset of coupling can lead to an inversion of the natural frequency difference and it may even induce desynchronization of two identical oscillators.

The outline of the paper is as follows: in Sec. II we review some basic properties of phase synchronization in two interacting oscillators. To illustrate the main ideas we study the route to synchronization in three different model systems (namely the Rössler system and an either chaotic or limit-cycle predator-prey oscillator), each having its own distinct frequency response to the onset of coupling. In Sec. III we demonstrate that the striking different synchronization properties in these models can be explained in terms of simple phase models. In Sec. IV we investigate the effects of asymmetric coupling, in Sec. V we explore the influence of noise and in Sec. VI we summarize our results. In the Appendix we give an explanation for the observed nonisochronicity of our models.

## II. ANOMALOUS PHASE SYNCHRONIZATION

We study the synchronization in a pair of asymmetrically coupled nonidentical oscillators

 

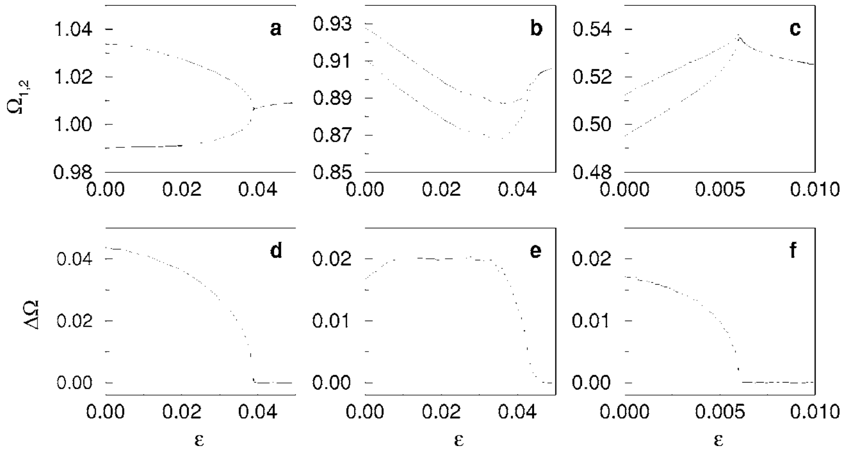


FIG. 1. Route to phase synchronization in two symmetrically coupled oscillators. Plotted are the observed frequencies of the individual oscillators $\Omega_{1,2}(\epsilon)$ (top) and the frequency difference $\Delta\Omega(\epsilon)$ (bottom) as a function of coupling strength $\epsilon$. (Left) Two coupled Rössler systems (3); (middle) two coupled chaotic food web models (5); (right) two coupled limit-cycle predator-prey models (6). Parameter values in all three cases: $b_2=1.0$ and $b_1=0.96$.

$$\dot{\mathbf{x}}_\mathbf{i} = \mathbf{F}(\mathbf{x}_\mathbf{i};b_i) + \epsilon_i C(\mathbf{x}_\mathbf{j} - \mathbf{x}_\mathbf{i}) + \eta_i(t) \quad (i,j=1,2 \quad i \neq j). \tag{1}$$

Here, in the absence of coupling each oscillator, $\mathbf{x}_i \in \mathbb{R}^n$, follows its own dynamics $\dot{\mathbf{x}}_i = F(\mathbf{x}_i, b_i)$. The oscillators are assumed to be nonidentical which is achieved by assigning to each oscillator $i$ an independent value of its control parameter $b_i$. We always assume that each oscillator is parameterized either on a limit cycle or on a regime with phase coherent chaos. Accordingly every, possibly chaotic, oscillator is characterized by a well defined natural frequency $\omega_i$. The oscillators are coupled with strength $\epsilon_i$, where we explicitely allow for asymmetrical coupling, $\epsilon_1 \neq \epsilon_2$. $C=\mathrm{diag}(c_1, c_2 \ldots, c_n)$ is a diagonal matrix which indicates the strength of the interaction in each component of the state vector $\mathbf{x}_i$. Further, in Sec. V we also allow for the possibility that both oscillators are under the influence of uncorrelated noise, $\eta_i(t)$ taken from a Gaussian distribution with zero mean and standard deviation $\sigma$, i.e., $\langle \eta_i(t)\eta_j(s)\rangle = \sigma^2\delta(t-s)\delta_{ij}$.

In general, the natural, unperturbed frequency of each oscillator will be a function of its control parameter, $\omega_i=\omega(b_i)$, which leads to a natural frequency mismatch $\Delta\omega=\omega_2-\omega_1$ between the two oscillators. Synchronization arises as an interplay between the interaction $\epsilon_i$ and the frequency mismatch $\Delta\omega$. Thereby, with the onset of coupling the frequency of each oscillator will be detuned,

$$\Omega_i = \Omega_i(\epsilon_i). \tag{2}$$

Here, we denote the observed oscillator frequency in the presence of coupling with a capital $\Omega_i(\epsilon_i)$ in contrast to the natural frequency $\omega_i$ of the uncoupled oscillator, i.e., $\omega_i=\Omega_i(0)$. Phase synchronization refers to the fact that with sufficient coupling strength $\epsilon_i > \epsilon_c$ the two oscillators rotate with the same frequency, $\Omega_1=\Omega_2$.

It is long known that phase synchronization arises naturally in two interacting limit cycle systems [3], but it is also possible in two coupled phase coherent chaotic oscillators by maintaining chaotic amplitudes [5]. Take for example two coupled Rössler systems [26]

$$\dot{x}_{1,2} = -b_{1,2}y_{1,2} - z_{1,2},$$

$$\dot{y}_{1,2} = b_{1,2}x_{1,2} + 0.15y_{1,2} + \epsilon_{1,2}(y_{2,1} - y_{1,2}),$$

$$\dot{z}_{1,2} = 0.4 + (x_{1,2} - 8.5)z_{1,2}. \tag{3}$$

In the parameter range $b_i \approx 1$ each oscillator shows phase coherent dynamics. Consequently a phase can easily be defined in the chaotic systems [3,5]. In this paper, we always estimate the phase as an angle in the $(x,y)$-phase plane, i.e., we measure the time averages $\bar{x}_i$ and $\bar{y}_i$ and define the phase as the angle

$$\theta_i(t) = \arctan\frac{y_i - \bar{y}_i}{x_i - \bar{x}_i}, \tag{4}$$

which is further unwrapped from the interval $[0\cdots 2\pi]$ into the real numbers, so that $\theta_i(t)$ is a time continuous function. The rotation frequency is then given as the long time average of phase velocity, $\Omega_i = \overline{\dot{\theta}_i(t)}$.

The path to synchronization in system (3) is depicted in Figs. 1(a) and 1(d). Both oscillators are diffusively coupled in the $y$-variable with equal strength $\epsilon_{1,2}=\epsilon$. Note, that this is a specific system of type (1) with the restriction of symmetric coupling, $\epsilon_1=\epsilon_2$, and vanishing noise, $\sigma=0$. Similar to [5] we study a small parameter mismatch between the two oscillators of $(b_2-b_1)/b_2=4\%$. As can be seen in Fig. 1(a) despite the chaotic amplitudes the transition to the synchronized state is very smooth. Both oscillators start out with a natural frequency difference $\Delta\omega=\omega_2-\omega_1$. With the onset of interaction both oscillator frequencies are detuned (2) and are attracted towards each other. Finally, at the critical coupling strength $\epsilon_c \approx 0.04$ they collide to a single frequency. This process can be visualized through a plot of the frequency difference $\Delta\Omega(\epsilon)$, which is a monotonically decreasing function of coupling strength [Fig. 1(d)]. When the coupling exceeds the critical value, $\epsilon > \epsilon_c$, the frequency difference disappears $\Delta\Omega(\epsilon)=0$ and the oscillators are synchronized in phase [5].

The question arises whether the simple route to synchronization as exemplified in Figs. 1(a) and 1(d) is general, e.g., whether the frequency disorder $\Delta\Omega(\epsilon)$ is always a monotonically decreasing function of coupling strength. To explore this case we first study two interacting chaotic predator-prey systems [9,14]

$$\dot{x}_{1,2} = a(x_{1,2} - x_0) - \alpha x_{1,2} y_{1,2},$$

$$\dot{y}_{1,2} = - b_{1,2} y_{1,2} + \alpha x_{1,2} y_{1,2} - \beta y_{1,2} z_{1,2} + \epsilon_{1,2}(y_{2,1} - y_{1,2}),$$

$$\dot{z}_{1,2} = - c(z_{1,2} - z_0) + \beta y_{1,2} z_{1,2}. \tag{5}$$

Each model describes a three trophic food chain where the basal species $x_i$ is consumed by the predator $y_i$ which itself is preyed upon by the top predator $z_i$. In the absence of interspecific interactions the dynamics is linearly expanded around the steady state $(x_0, 0, z_0)$ with coefficients $a$, $b_i$ and $c$. Predator-prey interactions are introduced via mass-action terms with strength $\alpha$ and $\beta$. Parameter values are taken as in [9,14] ($a=1, c=10, x_0=1.5, z_0=0.01, \alpha=0.1, \beta=0.6$). In this parameter range the model shows phase coherent chaotic dynamics, where the trajectory rotates with nearly constant frequency in the $(x_i, y_i)$-plane but with chaotic dynamics that appear as irregular spikes in the top predator $z_i$. This behavior of the food web model is reminiscent to the Rössler system (3) and therefore one might expect similar synchronization properties in both systems.

In Figs. 1(b) and 1(e) we study the synchronization of the two food web systems (5) which are interacting in the $y$-variable with symmetrical coupling $\epsilon_i = \epsilon$. Both oscillators vary in the value of their respective consumer death rates $b_i$, where we have used exactly the same parameter mismatch as in the two coupled Rössler systems. Despite the fact that both, Rössler and food web systems, have very similar attractor topology we find large differences in their response to the interaction. In the two coupled food web models with increasing coupling strength $\epsilon$ the observed frequencies of both oscillators are largely reduced [Fig. 1(b)], where the detuning of frequencies is a nearly linear function of $\epsilon$. Furthermore, the amount of frequency detuning is different for both oscillators. In the range of small coupling strength this effectively leads to an enlargement of the frequency difference $\Delta\Omega$ with increasing $\epsilon$. Only for larger values of coupling strength, $\epsilon > 0.03$, the frequencies are attracted towards each other, finally giving rise to phase synchronization. Thus, whereas in the two coupled Rössler systems the onset of the interaction leads to a monotonic decrease of $\Delta\Omega(\epsilon)$, in the two food web models the frequency difference is first amplified, with a maximal decoherence for intermediate levels of coupling.

This unusual increase of frequency difference with coupling strength has been denoted as anomalous phase synchronization [19,20]. In the two symmetrically coupled food web models (5) the amount of anomalous synchronization, measured as the maximal amplification of the natural frequency difference, is relatively small. However, as will be shown below, anomalous effects can be drastic enhanced.

As a third model example we investigate the synchronization in two limit-cycle predator-prey models [29]

$$\dot{x}_{1,2} = a x_{1,2}\left(1 - \frac{x_{1,2}}{K}\right) - \alpha \frac{x_{1,2} y_{1,2}}{1 + \kappa x_{1,2}},$$

$$\dot{y}_{1,2} = - b_{1,2} y_{1,2} + \alpha \frac{x_{1,2} y_{1,2}}{1 + \kappa x_{1,2}} + \epsilon_{1,2}(y_{2,1} - y_{1,2}). \tag{6}$$

Here, $x_i$ denotes the prey and $y_i$ the predator species, $a$ and $b_i$ are the birth and death rates, $K$ is the prey carrying capacity, $\alpha$ the predation rate and $\kappa$ the half saturation constant of the functional response (parameter values $a=1$, $\alpha=3$, $K=3$, $\kappa=1$). For sufficient large values of enrichment $K$, the predator-prey system (6) is well known to exhibit limit cycle oscillations. The path to synchronization in the interacting oscillators ($\epsilon_{1,2} = \epsilon$) is shown in Figs. 1(c) and 1(f). Again the observed frequencies $\Omega_i(\epsilon)$ are nearly linearly detuned with coupling strength, but now the frequencies are increasing functions of $\epsilon$. Thus, in the coupled limit cycle models (6) the onset of coupling has exactly the opposite effect as in the two chaotic food web models (5). Phase synchronization, however, sets in smooth and the shape of $\Delta\Omega(\epsilon)$ is comparable to that of the Rössler system Fig. 1(d).

## III. PHASE EQUATIONS

The numerical observations of the previous section can be explained with the help of simple phase models, which describe the synchronization properties of system (1) in the limit of weak interaction [2,6,16]

$$\dot{\theta}_i = \omega_i + \epsilon_i \Gamma_i(\theta_j - \theta_i) + \eta_i(t) \quad (i,j = 1,2 \quad i \neq j). \tag{7}$$

Here, the state of each oscillator $i$ is described solely in terms of its phase $\theta_i$. In the absence of coupling the phase is assumed to rotate uniformly according to the oscillators natural frequency, $\dot{\theta}_i = \omega_i$. The effects of coupling are represented by the interaction function $\Gamma_i$ which, in general, is a $2\pi$-periodic function of the phase difference, $\phi = \theta_2 - \theta_1$. Further, in system (7) also the possibility of additional additive noise $\eta_i$ has been taken into account.

This phase-description of two weakly interacting limit cycle oscillators freely translates to the case of phase-coherent chaotic oscillations. According to [5] the phase dynamics of a single autonomous chaotic oscillator can be described as $\dot{\theta}_i = \omega_i + F_i(A_i)$, where $F_i(A_i)$ accounts for the dependence of the instantaneous observed frequency on the chaotic amplitude of oscillation $A_i(t)$. For two coupled oscillators this generalizes to [5]

$$\dot{\theta}_i = \omega_i + F_i(A_i) + \epsilon_i \Gamma_i(\theta_j - \theta_i). \tag{8}$$

In practice it has been shown [5] that the chaotic force $F_i(A_i)$ can be considered as effective noise. In this approximation the phase dynamics of the chaotic systems, Eq. (8), effectively is reduced to that of two noisy oscillators, Eq. (7). Therefore we can treat the synchronization of both, chaotic and nonchaotic systems, in a similar way. For the moment, however, we neglect the effects of noise and restrict the analysis to the deterministic system.

The synchronization properties of system (7) are determined by the phase difference $\phi = \theta_2 - \theta_1$. Subtraction of both equations in (7) leads to a single equation for the evolution of $\phi$ [in the deterministic case, i.e., setting $\eta_i(t) = 0$]

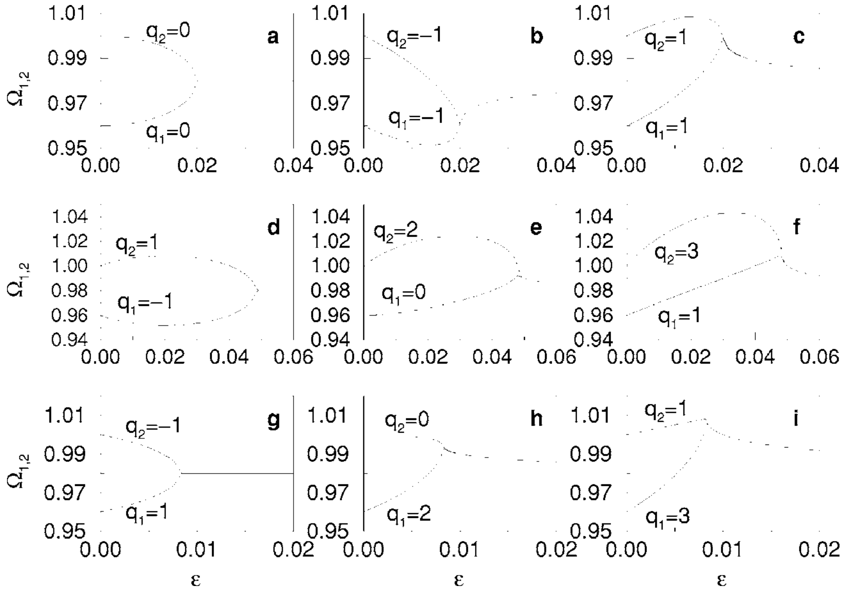


FIG. 2. Path to synchronization in two symmetrically coupled nonisochronous phase oscillators (15). Plotted are the observed frequencies of the individual oscillators, $\Omega_{1,2}(\epsilon)$, as a function of coupling strength $\epsilon$ for various combinations of nonisochronicities $q_{1,2}$. Top [(a),(b),(c)]: usual path to synchronization $\Delta q=0$; middle [(d),(e),(f)]: anomalous desynchronization, $\Delta q=2$; bottom [(g),(h),(i)]: anomalous synchronization enhancement, $\Delta q=-2$. Natural frequencies are $\omega_2=1$ and $\omega_1=0.96$.

$$\dot{\phi}=\Delta\omega+[\epsilon_2\Gamma_2(-\phi)-\epsilon_1\Gamma_1(\phi)]. \tag{9}$$

In order to proceed the interaction function $\Gamma_i(\phi)$ has to be specified. In general, it is possible to calculate the interaction function from the original system (1) [2]. However, for most practical purposes it is more convenient to approximate $\Gamma$ by simple conceptional functions. Usually it is assumed that the interaction disappears for $\phi=0$ when the two oscillators are in an identical state, $\Gamma_i(0)=0$. The simplest $2\pi$-periodic function with this property is given by the sine-function

$$\Gamma_i(\phi)=\sin(\phi), \tag{10}$$

which provides a canonical description for the synchronization of two oscillators (note that stability of the synchronized solution requires that $\Gamma_i'(0)>0$). Inserting this into Eq. (9) and further assuming symmetrical coupling, $\epsilon_1=\epsilon_2=\epsilon$, we obtain for the dynamics of the phase difference, $\dot{\phi}=\Delta\omega-2\epsilon\sin(\phi)$. To obtain the coupling modified frequency difference $\Delta\Omega(\epsilon)$ we calculate the time-averaged "beating period" $T=\int_0^{2\pi}(d\phi/\dot{\phi})$ which can easily be integrated (see for example [27]) and leads to the well known result for the frequency difference, $\Delta\Omega=2\pi/T$, in two coupled phase oscillators ($\epsilon<\Delta\omega/2$)

$$\Delta\Omega(\epsilon)=\sqrt{\Delta\omega^2-4\epsilon^2}. \tag{11}$$

On the other hand, by adding up both equations in (7) we find that for the simple model (10) the mean frequency is not affected by coupling, $\dot{\theta}_1+\dot{\theta}_2=\omega_1+\omega_2=\text{const}$. Therefore, the coupling modified frequency of each oscillator can be written as

$$\Omega_{2,1}(\epsilon)=\frac{\omega_1+\omega_2}{2}\pm\frac{1}{2}\Delta\Omega(\epsilon). \tag{12}$$

This dependence on coupling strength as described by Eqs. (11) and (12) is plotted in Fig. 2(a). Due to the onset of coupling the two observed frequencies are smoothly attracted towards each other according to the square root law (11), while the mean frequency stays constant.

By comparison with Fig. 1(a) it becomes immediately apparent that the simple phase model (10) gives an excellent description for the path to synchronization in the two Rössler systems (3). However, it fails to describe the linear detuning for small $\epsilon$ that is observed in the two investigated ecological models [see Figs. 1(b) and 1(c)]. This is also evident from a Taylor expansion of the frequencies $\Omega_i(\epsilon)$ (12)

$$\Omega_{1,2}(\epsilon)=\omega_{1,2}\pm\frac{\epsilon^2}{\Delta\omega}+O(\epsilon^3). \tag{13}$$

Here, the slope of $\Omega_i(\epsilon)$ as a function of $\epsilon$ must always be zero for small coupling, $\lim_{\epsilon\to 0}[d\Omega_i(\epsilon)/d\epsilon]=0$, which is in contrast to the numerical observations in the two ecological models.

The phase description can be improved by using more realistic interaction functions. Developing $\Gamma(\phi)$ into a Fourier series, with the additional constraint that $\Gamma(0)=0$, leads in first order to the following generalization of the simple phase model (10):

$$\Gamma_i(\phi)=\sin(\phi)+q_i(1-\cos\phi). \tag{14}$$

In this equation the parameter $q_i$ is known as the nonisochronicity of oscillation [2,19,20] and describes the amplitude dependence of the rotation frequency (see Appendix). We now investigate the mutual entrainment of two nonidentical phase oscillators which are coupled according to (14)

$$\dot{\theta}_1=\omega_1+\epsilon_1[\sin(\phi)+q_1(1-\cos\phi)],$$

$$\dot{\theta}_2=\omega_2+\epsilon_2[\sin(-\phi)+q_2(1-\cos\phi)]. \tag{15}$$

Figure 2 shows the results of the numerical simulations of system (15) for different parameter combinations of nonisochronicities $q_{1,2}$, where again we first assume symmetrical coupling strength $\epsilon_1=\epsilon_2=\epsilon$. Obviously, for nonvanishing values of $q_i$ the onset of synchronization can be drastically

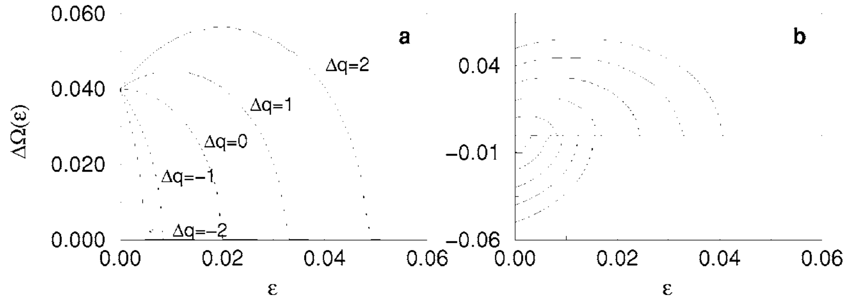


FIG. 3. Anomalous synchronization in two symmetrically coupled nonisochronous phase oscillators (15). (a) Natural frequency mismatch is fixed, $\Delta\omega=0.04$ and the difference of nonisochronicity varies in the range from $\Delta q=2$ to $\Delta q=-2$. (b) Difference in nonisochronicity is fixed, $\Delta q=1$, and $\Delta\omega$ is varied between $\Delta\omega=0.05\cdots-0.05$ from top to bottom.

modified. In all parameter combinations for large $\epsilon$ the two oscillators eventually become synchronized in phase. However, superimposed to this usual route to synchronization is a frequency detuning in the range of small coupling. From Fig. 2 this detuning for each oscillator is strongly correlated to its value $q_i$.

As an explanation we follow the arguments in [19,20]. Assume that for small coupling levels, $\epsilon \ll \epsilon_c$, the oscillators (15) are rotating nearly independently. After time averaging, $\langle \sin\phi\rangle \approx \langle \cos\phi\rangle \approx 0$, we are left with the following uncoupled equations for the mean phase evolution

$$\Omega_i(\epsilon) = \omega_i + q_i\epsilon + O(\epsilon^2). \tag{16}$$

From this expression (16) it is clear that in the range of small coupling the frequencies $\Omega_i(\epsilon)$ are linearly detuned, where the slope is exactly given by the nonisochronicity in each oscillator

$$q_i = \lim_{\epsilon\to 0}\frac{d\Omega_i(\epsilon)}{d\epsilon}. \tag{17}$$

Now recall that the oscillators differ in their respective values of $\omega_i$ and $q_i$. Subtracting both equation in (16) we obtain for the difference of the observed frequencies up to first order in $\epsilon(\Delta q = q_2 - q_1)$

$$\Delta\Omega(\epsilon) = \Delta\omega + \epsilon\Delta q + O(\epsilon^2). \tag{18}$$

Thus, if $\Delta\omega>0$ the frequency difference $\Delta\Omega(\epsilon)$ is an increasing function of coupling strength when $\Delta q>0$. This is the origin of anomalous synchronization. In the reverse situation, when $\Delta q<0$ the frequency difference is reduced by coupling, resulting in an anomalous enhancement of synchronization.

These results, which are valid only in the range of small coupling, are confirmed by an exact calculation in the whole coupling range. Inserting the interaction function (14) into (9) we find for the phase difference

$$\dot\phi = \Delta\omega - \epsilon[2\sin\phi + \Delta q(\cos\phi - 1)]. \tag{19}$$

Then, in generalization to Eq. (11) after straightforward integration the mean frequency difference follows:

$$\Delta\Omega(\epsilon) = \sqrt{\Delta\omega^2 + 2\epsilon\Delta\omega\Delta q - 4\epsilon^2}. \tag{20}$$

In this expression, the synchronization characteristics are determined by the product $\Delta\omega\Delta q$. If $\Delta\omega\Delta q>0$ then small coupling tends to desynchronize the oscillators and we observe anomalous phase desynchronization. In contrast, negative values of $\Delta\omega\Delta q$ lead to an anomalous enhancement of synchronization.

This is also demonstrated in Fig. 3(a) where the coupling modified frequency difference $\Delta\Omega(\epsilon)$ of system (15) has been plotted for the case of a positive frequency mismatch $\Delta\omega=0.04$. If $\Delta q>0$ we observe anomalous desynchronization whereas for $\Delta q<0$ synchronization is enhanced. For comparison, in Fig. 3(b) the difference in nonisochronicity is held fixed, $\Delta q=1$, and $\Delta\Omega(\epsilon)$ is plotted for different values of $\Delta\omega$. We find anomalous desynchronization for $\Delta\omega>0$, whereas $\Delta\omega<0$ leads to enhanced synchronization.

We want to stress, that Eq. (17) provides a convenient way to estimate the amount of nonisochronicity in any given model, even when the dynamics are chaotic. The idea is simply to introduce interaction between two models and to measure the change of the rotation frequency as a function of coupling strength, $\Omega_i(\epsilon)$. The nonisochronicity is then estimated as the linear response of the coupling modified frequency, Eq. (17). Applied to Fig. 1, for example, we find that the nonisochronicity of the chaotic food web model (5) is negative ($q_1=-1.7\pm0.1, q_2=-1.43\pm0.05$), whereas in the limit-cycle predator-prey model (6) the $q_i$ are positive ($q_1=5\pm0.1, q_2=4\pm0.1$). In comparison, in the Rössler system (3) nonisochronicity is rather small ($q_1=0.15\pm0.1, q_2=0.1\pm0.1$). As a result we estimate a positve mismatch of nonisochronicity, $\Delta q=0.27$, for the chaotic food web model (5), a large negative value $\Delta q=-1$ for the limit cycle model (6), and a mismatch close to zero $\Delta q\approx-0.05$ for the Rössler system (3). Now taking into account that in all three model systems considered in Fig. 1 we have $\Delta\omega>0$ these results provide an explanation for the strikingly different routes to synchronization which are exhibited by these models.

## IV. ANOMALOUS SYNCHRONIZATION WITH ASYMMETRIC COUPLING

In the previous section we have discussed how anomalous effects can arise when the two oscillators differ in their re-

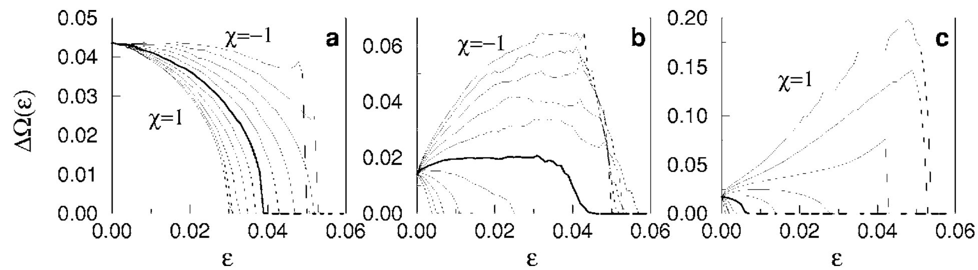


FIG. 4. Anomalous phase synchronization in two asymmetrically coupled oscillators. Plotted is the observed frequency difference $\Delta\Omega(\epsilon)$ as a function of coupling strength $\epsilon$ for different values of the asymmetry parameter $\chi$. (Left) Two coupled Rössler systems (3); (middle) two coupled chaotic food web models (5); (right) two coupled limit-cycle oscillators (6). The results for symmetrical coupling $\chi=0$ are indicated as thick lines (compare to Fig. 1).

spective value of nonisochronicity, $q_{1,2}$. However, even in systems with a large absolute value of nonisochronicity, such as the two ecological models (6), the difference of nonisochronicity usually is relatively small, i.e., $\Delta q/q_i \ll 1$. As a consequence, anomalous effects are rather small in such systems. In the following we show that anomalous synchronization appears even when the oscillators have nearly identical nonisochronicity if the coupling between the oscillators is asymmetric. Note that a perfect coupling symmetry, $\epsilon_1=\epsilon_2$, between two interacting dynamic units must be considered to be an exception. Instead, in real systems the coupling strength usually will be different in both directions, $\epsilon_1 \neq \epsilon_2$. Asymmetric coupling can arise for various reasons. One extreme case is unidirectional coupling, where one oscillator is driving the other system but itself remains unperturbed [28]. However, also more moderate forms of coupling asymmetry frequently arise, for example due to different size or mass of the oscillators [21,34]. Consider again two oscillators (15) which are coupled with strength

$$\epsilon_{1,2} = \epsilon(1 \mp \chi). \tag{21}$$

Here the parameter $\chi=(\epsilon_2-\epsilon_1)/(\epsilon_1+\epsilon_2)$ determines the coupling asymmetry, $-1 \leqslant \chi \leqslant 1$. The extreme cases $\chi=\pm 1$ refer to unidirectional coupling, whereas $\chi=0$ reduces to symmetrical coupling. In analogy to (19) we find for the phase difference

$$\dot{\phi} = \Delta\omega - \epsilon[2 \sin \phi + Q(\cos \phi - 1)], \tag{22}$$

with the effective parameter

$$Q = \frac{\epsilon_2 q_2 - \epsilon_1 q_1}{\epsilon} = \Delta q + 2\chi q. \tag{23}$$

Here, we have used the notation $\epsilon=(\epsilon_1+\epsilon_2)/2$ and $q=(q_1+q_2)/2$. Interesting are the two limiting cases. If $\epsilon_1=\epsilon_2$ (symmetric coupling) then $Q$ reduces to the difference of nonisochronicities $Q=\Delta q$. In the other extreme of identical nonisochronicities $q_1=q_2$ the effective parameter is given by $Q=2\chi q=q\frac{\Delta\epsilon}{\epsilon}$. Therefore, nonvanishing values of the effective parameter $Q$, with the consequence of anomalous effects, can be achieved even if the two oscillators have identical nonisochronicities, $\Delta q=0$.

Proceeding as in the previous section the observed frequency difference yields

$$\Delta\Omega(\epsilon) = \sqrt{\Delta\omega^2 + 2\epsilon Q \Delta\omega - 4\epsilon^2}, \tag{24}$$

and by comparison with (20) anomalous enlargement arises if the product $Q\Delta\omega > 0$ and anomalous synchronization enhancement if $Q\Delta\omega < 0$. The synchronization threshold of the asymmetrically coupled oscillators is given by

$$\epsilon_c = \frac{\Delta\omega}{4}[Q + \sqrt{Q^2+4}]. \tag{25}$$

Obviously, $\epsilon_c$ is monotonously increasing with $Q$. If $Q=0$ this reduces to the well-known nonisochronous case $\epsilon_c=\Delta\omega/2$. For large values of the effective parameter, $Q \gg 1$, the synchronization threshold grows linearly with $Q$, $\epsilon_c \approx \frac{\Delta\omega}{2}Q$, whereas for $Q \ll -1$ the threshold goes to zero, $\epsilon_c \approx \frac{-\Delta\omega}{2Q}$.

In Fig. 4 we investigate the influence of coupling asymmetry in the three model systems of Sec. II by calculating $\Delta\Omega(\epsilon)$ for different values of the asymmetry parameter $\chi$. In the Rössler system, changing the coupling asymmetry $\chi$ has not much influence on the path to synchronization because in this system the absolute value of $q$ is very small. In contrast, in the two, chaotic or limit cycle, food web models nonisochronicity is much larger and by varying $\chi$ we observe drastic changes in the form of $\Delta\Omega(\epsilon)$. In fact, the path to synchronization can be totally reversed from enhanced synchronization to anomalous desynchronization. In the chaotic food web model a negative value of $\chi$ is necessary to achieve anomalous desynchronization, $Q\Delta\omega > 0$, since in this model the sign of the $q_i$ is negative. In contrast, in the limit-cycle predator-prey model (6) nonisochronicity is positive, $q_i > 0$, and anomalous desynchronization is achieved for positive $\chi$, whereas negative $\chi$ lead to anomalous enhancement of synchrony.

As shown in Fig. 4 for $\chi \neq 0$ the amount of anomalous desynchronization can be very large. For example, in the two ecological models for coupling $\epsilon$ just smaller than the synchronization threshold $\Delta\Omega$ can be several times as large as the natural frequency difference $\Delta\omega$. Also the synchronization threshold rises very fast with the coupling asymmetry $\chi$. Indeed in the predator-prey model (6) for $\chi > 0.8$ the threshold becomes $\epsilon_c > 0.05$, which is about ten times as large as the synchronization threshold of the symmetrically coupled oscillators (see Fig. 1). To explain this strong increase of the synchronization threshold, we roughly estimate the magni-

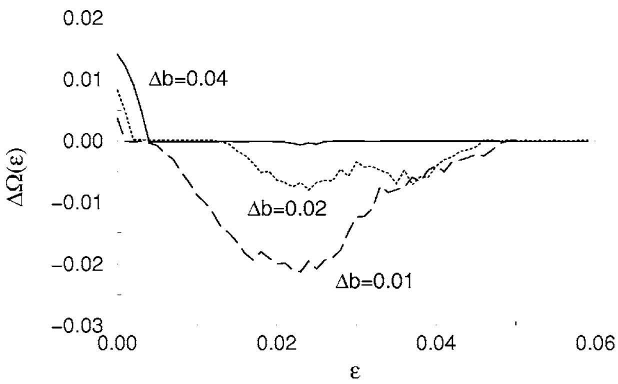


FIG. 5. Anomalous inversion of the natural frequency difference in two unidirectionally ($\chi=1$) coupled chaotic food web models (5). Plotted is the observed frequency difference $\Delta\Omega(\epsilon)$ as a function of coupling strength $\epsilon$. Parameter values $b_{1,2}=b_0\mp 0.5\Delta b$ with $b_0=0.98$ and $\Delta b=0.04$ (solid line), $\Delta b=0.02$ (dotted line), and $\Delta b=0.01$ (dashed line). Even though $\Delta\omega>0$ the coupling modified frequency difference can become negative, $\Delta\Omega<0$.

tude of the effective parameter $Q$. From Eq. (23) we know that $Q=2\chi q$ if $\Delta q$ can be neglected. Since in the model (6) we estimated $q\approx 5$, the effective parameter can be as large as $Q\approx 10\chi$. Now, from Eq. (25) for large $Q$ we have $\epsilon_c(Q)\approx\epsilon_c(0)Q\approx\epsilon_c(0)10\chi$.

All these numerical observations are in perfect agreement to our theory (24) and clearly demonstrate that anomalous effects can strongly be enhanced in the presence of asymmetrical coupling schemes. However, so far there remain two basic restrictions. First, up to now anomalous effects essentially depend on the fact that the two oscillators are nonidentical. Even though coupling is able to enlarge an existing frequency difference, it is not possible in this way to desynchronize two identical oscillators. This is obvious for example from Eq. (25), which implies that for $\Delta\omega=0$ the synchronization threshold disappears, $\epsilon_c=0$. Secondly, even though anomalous effects can considerably modify the usual form of the function $\Delta\Omega(\epsilon)$, so far they cannot in principle invert the sign of the natural frequency mismatch, i.e., if $\Delta\omega>0$ this implies that for all coupling values also $\Delta\Omega(\epsilon)\geqslant 0$.

In the following we will show that these two restriction do not apply in chaotic or noisy systems. Take for example, the two interacting chaotic food web models (5) with an asymmetry parameter $\chi>0$, i.e. for very strong synchronization enhancement. In this case we find a small region with an inversion of the frequency mismatch, $\Delta\Omega(\epsilon)<0$. In Fig. 4(b), for our usual parametrization this effect is very small and can only be observed numerically. Therefore, in Fig. 5 we have plotted $\Delta\Omega(\epsilon)$ in the case of unidirectional coupling, $\chi=1$, for different values of the parameter mismatch. Using our standard parameters, $\Delta b=0.04$, we find enhanced synchronization in agreement to (24). However, new effects appear for smaller values of $\Delta b$. Then the function $\Delta\Omega(\epsilon)$ with increasing coupling at first is sharply reduced and for $\epsilon>\epsilon_c$ the two oscillators are synchronized in phase $\Delta\Omega=0$. However, by increasing $\epsilon$ even further, this region is followed by a second desynchronization regime, which is characterized by a negative sign of $\Delta\Omega$.

We denote this unusual effect, that the frequency difference of the interacting oscillators has the opposite sign as that of the uncoupled oscillators, as anomalous frequency inversion. The effect arises only together with anomalous synchronization enhancement (here $\chi>0$), however not only as in Fig. 5 for the extreme case of unidirectional coupling, but also for more moderate coupling asymmetry. Further, the effect is as stronger pronounced the more identical the two oscillators are parameterized. In our simulations we find that anomalous frequency inversion is restricted to chaotic systems and cannot be observed in deterministic limit cycle oscillators. In terms of our theory, based on deterministic phase oscillators, this inversion of $\Delta\Omega$ cannot be explained. However, recall that the phase reduction of a chaotic oscillator (8) effectively leads to a noisy system. In the following we show that anomalous frequency inversion naturally emerges after the inclusion of noise into the phase models with nonisochronicity.

## V. ANOMALOUS PHASE SYNCHRONIZATION IN THE PRESENCE OF NOISE

In this section we investigate the interplay between anomalous synchronization and additive white noise. To these ends we introduce additive Gaussian noise $\eta_{1,2}(t)$ of strength $\sigma_{1,2}$ into the phase model (7). Since the difference of two Gaussian noises is again Gaussian the basic equation for the phase difference (22) is modified to

$$\dot{\phi}=\Delta\omega-\epsilon[2\sin\phi+Q(\cos\phi-1)]+\eta(t). \qquad (26)$$

Here $\eta(t)$ is uncorrelated Gaussian noise of strength $\sigma^2=\sigma_1^2+\sigma_2^2$, i.e., $\langle\eta(t)\eta(s)\rangle=\sigma^2\delta(t-s)$.

In the isochronous case $Q=0$ the influence of noise is well known [3]. This is shown in Fig. 6(a) where we investigate the synchronization arising in two isochronous phase oscillators. Without noise $\Delta\Omega(\epsilon)$ is reduced according to the square root scaling (11). Under the influence of noise the observed frequency difference $\Delta\Omega(\epsilon,\sigma)$ is still a monotonically decreasing function of coupling strength $\epsilon$, but can never become zero due to noise induced phase slips. Thus,

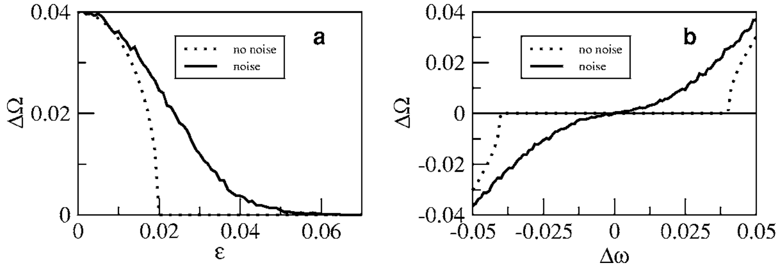


FIG. 6. Phase synchronization in the presence of noise. Plotted is the frequency difference $\Delta\Omega$ of two coupled isochronous ($q_i=0$) phase oscillators (7) with interaction (10) without noise, $\sigma=0$, (dotted line) and with noise, $\sigma=0.15$ (solid line). (Left) $\Delta\Omega$ as a function of coupling strength $\epsilon(\Delta\omega=0.04)$. (Right) $\Delta\Omega$ as a function of the frequency mismatch $\Delta\omega(\epsilon=0.02)$.

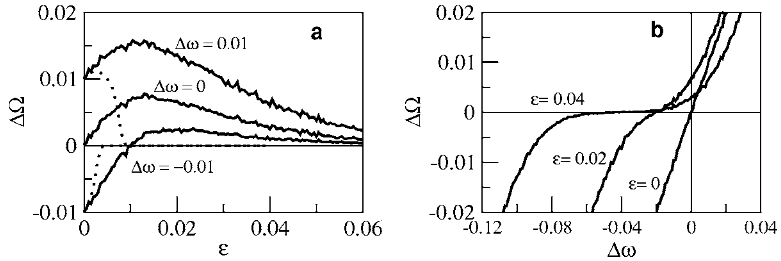


FIG. 7. Anomalous synchronization in the presence of noise. Plotted is the frequency difference $\Delta\Omega$ in two asymmetrically coupled nonisochronous phase oscillators (15) with $Q=1(\chi=0.5,q_i=1)$ under the influence of additive Gaussian noise with strength $\sigma=0.15$ (solid lines). (Left) $\Delta\Omega$ as a function of coupling strength $\epsilon$ for different values of $\Delta\omega$. Further plotted are results without noise $\sigma=0$ (dotted lines). (Right) $\Delta\Omega$ as a function of the natural frequency mismatch $\Delta\omega$ for different values of $\epsilon$.

we observe a sigmoidal reduction of the frequency difference with increase of $\epsilon$. Similar effects can be seen in a plot of the observed frequency difference as a function of the natural frequency difference, $\Delta\Omega(\Delta\omega)$ [see Fig. 6(b)]. Without noise, for small $|\Delta\omega|$ we observe a locking plateau in which $\Delta\Omega=0$. With the introduction of noise this curve is smoothed out, however the plateau for small $\Delta\omega$ is still visible.

Here now, we are interested in the influence of noise when the oscillators have nonzero value of nonisochronicity, $q_i \neq 0$. This is depicted in Fig. 7, where the synchronization in two asymmetrically coupled phase oscillators (15) is studied. Both oscillators have identical nonisochronicity $q_{1,2}=1$ and the asymmetry parameter equals $\chi=0.5$ (by setting $\epsilon_2=3\epsilon_1$) so that the effective parameter $Q=2q\chi=1$. In Fig. 7a we plot the coupling dependence of the frequency difference, $\Delta\Omega(\epsilon)$. In the noiseless case (dotted lines) the results correspond to Fig. 3(b), i.e., if $\Delta\omega>0$ we observe anomalous desynchronization, whereas for $\Delta\omega<0$ synchronization is anomalously enhanced. With the inclusion of noise of moderate strength, $\sigma=0.15$, these transitions are largely modified [Fig. 7(a), solid lines]. Then, for positive frequency mismatch, $\Delta\omega=0.01$, the anomalous effects are more pronounced and the desynchronization is considerably enhanced. On the other hand, if $\Delta\omega=-0.01$ with increase of coupling strength the curve $\Delta\Omega(\epsilon)$ crosses the x axis, giving rise to an anomalous inversion of the natural frequency difference. Interesting effects arise also when the oscillators are identical and the natural frequency mismatch is zero, $\Delta\omega=0$. Without noise in this case the oscillators remain synchronized for all coupling levels, $\Delta\Omega(\epsilon)=0$. However, as demonstrated in Fig. 7(a) under the influence of noise small coupling is able to desynchronize the two identical oscillators.

In Fig. 7(b) we plot $\Delta\Omega(\Delta\omega)$ for the two nonisochronous oscillators ($Q=1$). Without coupling, $\epsilon=0$, the noise has no effect and $\Delta\Omega=\Delta\omega$. With the introduction of coupling a locking plateau becomes visible. With increasing levels of coupling strength the plateau becomes more pronounced, however additionally, the location of the plateau is shifted linearly with $\epsilon$. If $Q>0$ (as in Fig. 7) the plateau is shifted towards negative values of $\Delta\omega$, whereas for negative $Q$ the plateau is shifted in the other direction (not plotted). A similar shift of the locking region has been described for deterministic systems [20], but here we show that it also takes place in noisy systems. This has the consequence that with the inclusion of noise we can always find a natural frequency mismatch, exactly in the middle of the noiseless locking region $\Delta\omega_s=\epsilon Q$, for which the observed frequency difference disappears, $\Delta\Omega(\Delta\omega_s)=0$. Usually one would expect that frequency locking sets in for similar natural frequencies, i.e., $\Delta\omega_s=0$. However, here we find that this is different for noisy nonisochronous oscillators where frequency locking is initiated for oscillators with very different natural frequencies $\Delta\omega_s\neq 0$.

The anomalous inversion of the natural frequency mismatch between the two noisy phase oscillators in Fig. 7(a) is very similar to that which is observed in two asymmetrically coupled chaotic food web models in Fig. 5. As mentioned above, the phase reduction of two coupled chaotic oscillators (8) always contains an effective noise due to the erratic influence of the chaotic amplitudes. This noise, together with the nonisochronicity of the model and the asymmetry of coupling, is able to invert the natural frequency mismatch similar to the behavior in the noisy phase models in Fig. 7. However, there are also differences in the frequency inversion in the noisy phase model and the actual chaotic oscillators. Especially in Fig. 5 a synchronized plateau can be observed for a finite range of the coupling parameter before the frequeny inversion is achieved. This plateau, on the other hand, does not appear in the noisy phase model results. The reasons for this difference are partly explained by keeping in mind that the phase reduction of a chaotic oscillator (8) is only an approximation. Especially for different parameterization of the two oscillators one would expect that the chaotic amplitudes still play a role. This confirms with the fact that the plateaus in Fig. 5 reduce in size when the frequency inversion is large, so that for a large degree of frequency inversion chaotic oscillators and noisy phase oscillators behave very similar.

We want to stress that in the deterministic limit cycle system (6) similar effects such as anomalous frequency inversion or desynchronization of two identical oscillators are not possible. However, with the addition of noise we are able to induce similar routes to synchronization as exemplified in Fig. 7. This is tested in Fig. 8, where we investigate a system of two coupled limit-cycle models (6) with a moderate asymmetrical coupling of $\chi=0.75$. In Fig. 8(a) we plot the time evolution of the phase difference $\phi(t)$ for different values of coupling strength. We have adjusted a negative parameter difference $\Delta b<0$ so that the natural frequency mismatch is negative $\Delta\omega<0$. Without coupling, $\epsilon=0$, the phase difference is on average linearly decreasing with time, $\langle\dot{\phi}\rangle=\Delta\omega t$.

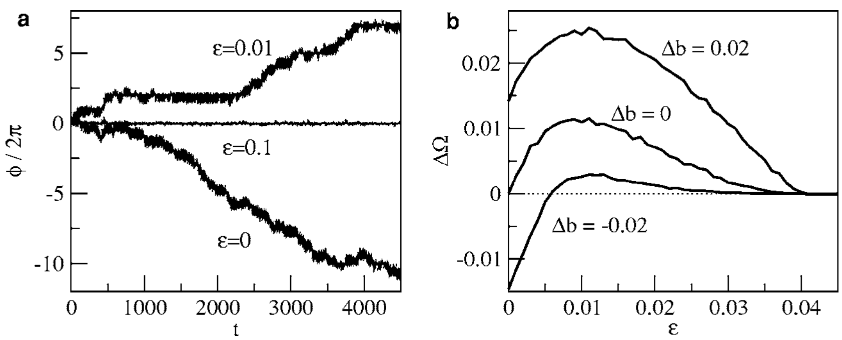


FIG. 8. Path to synchronization in two asymmetrically coupled food web models (6) in the presence of additive white noise. (Left) Phase evolution $\phi(t)/2\pi$ for different levels of coupling strength $\epsilon=0$, $\epsilon=0.01$, and $\epsilon=0.1$. Parameters $\chi=0.75$, $\Delta b=-0.02$, and $\sigma=0.005$. (Right) Frequency difference $\Delta\Omega(\epsilon)$ as a function of coupling strength. Parameter mismatch between both oscillators $b_{1,2}=0.98\mp0.5\Delta b$ with $\Delta b=-0.02$, $\Delta b=0$, and $\Delta b=0.02$. Other parameters $\sigma=0.01$ and $\chi=1/3$.

In this case the only effect of the noise is to give rise to small fluctuations of $\phi(t)$ around this linear decline. However, as shown in Fig. 8a, unusual results appear when coupling is switched on. For intermediate values of coupling strength, $\epsilon=0.01$, the slope of the phase evolution becomes positive, which clearly demonstrates the presence of anomalous frequency inversion. Interestingly, the time course of $\phi(t)$ is composed from a succession of discrete jumps of $2\pi$. This is an indication that the increase of the phase difference is connected to noise induced phase slips (see below). Due to these phase slips the phase difference $\phi(t)$ on average is an increasing function of time and the observed frequency difference $\Delta\Omega(\epsilon)$ becomes positive. Only for very large coupling strength synchronization sets in again and the slope of $\phi(t)$ reduces to zero.

More systematically this behavior is explored in Fig. 8(b) where the observed frequency difference is plotted as a function of coupling strength. Note, the qualitative similarity between Fig. 8(b) and the corresponding simulation of two noisy phase oscillators Fig. 7. Clearly, if $\Delta b<0$ then we observe anomalous frequency inversion and $\Delta\Omega(\epsilon)$ changes sign for intermediate values of $\epsilon$. If $\Delta b>0$ we observe anomalous desynchronization which is amplified in magnitude by the noise. Figure 8 also includes a simulation with two identical oscillators, $\Delta b=0$. Without noise then the oscillators would always be perfectly synchronized in phase, for all values of of the coupling strength $\epsilon$. However, in the noisy system with the onset of coupling both identical oscillators are rotating with nonidentical frequencies. Note, that this desynchronization of two identical oscillators with the onset of coupling requires a breaking of the exchange symmetry. Here, this symmetry breaking is caused by the asymmetry of the interaction.

At this point one remark of caution is in order. Already in the noiseless case, Fig. 4(c), we demonstrated that in asymmetrically interacting oscillators, $\chi\neq0$, the coupling threshold easily can be magnified about one order of magnitude compared to the case with symmetrical coupling. This has the consequence that in our numerical simulations even for very small parameter mismatch we are forced to explore regions with large coupling strength. This is even more so the case for noisy systems as shown for example in Fig. 8. At the same time, in the limit of large $\chi$ the system of two coupled oscillators corresponds to an externally driven system, which under sufficiently strong external forcing model (6) is well known to be able to exhibit chaotic dynamics with many complications such as multiple coexisting attractors [28]. In fact, in our simulations of the predator-prey model (6) for large values of $\chi$ we observe coupling induced chaotic dynamics, and it is not clear if our basic assumption of phase coherency is still valid. Even if the individual oscillators are rotating with a constant well defined frequency, the presence of chaotic dynamics easily can deform the phase portrait in such a way that our measurement of the phase as an angle in phase space, Eq. (4), does not apply any more. In the present study, we have always taken care to stay in a dynamic regime where the phase is well defined. However a naive use of formula (4) without checking the actual trajectories easily can give rise to spurious results.

In the following we provide an intuitive understanding of these numerical results. For this we make use of the fact that the evolution of the phase difference $\phi$ in Eq. (9) can be described in terms of a $2\pi$-periodic potential $V(\phi)$

$$\dot{\phi}=-\frac{d}{d\phi}V(\phi), \tag{27}$$

with

$$V(\phi)=-\Delta\omega\phi-\int d\phi[\epsilon_2\Gamma_2(-\phi)-\epsilon_1\Gamma_1(\phi)] \tag{28}$$

up to an arbitrary integration constant. The dynamics of the phase difference can be represented as that of an overdamped particle in the tilted periodic potential (28) [3,31–33]. In this picture, the effect of noise is to induce phase jumps between neighboring minima in the potential. A nonvanishing value of $\Delta\Omega$ corresponds to a directed transport (i.e., a finite particle current) in the potential.

Using our basic model (22) we can easily calculate the potential (28) for the case of two asymmetrically coupled nonisochronous oscillators (see Fig. 9)

$$V(\phi)=-(\Delta\omega+\epsilon Q)\phi-\epsilon(2\cos\phi-Q\sin\phi). \tag{29}$$

Without coupling, $\epsilon=0$, the potential (29) is a tilted straight line, $V(\phi)=-\Delta\omega\phi$, with a slope that is determined by the natural frequency mismatch $\Delta\omega$. This tilt is respon-

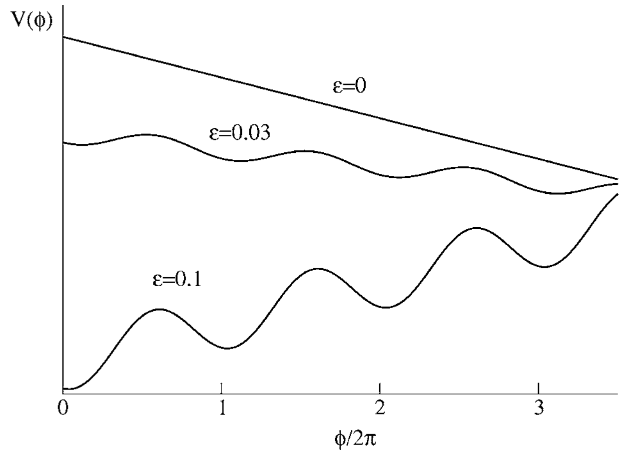


FIG. 9. Two different effects of coupling in nonisochronous oscillators. Plotted is the potential $V(\phi)$ (29) for three different levels of coupling strength. No coupling ($\epsilon=0$): the tilt of the potential is to the right. Intermediate coupling strength ($\epsilon=0.03$): the interaction gives rise to a periodic modulation of the potential; the tilt of the potential is still to the right. Strong coupling ($\epsilon=0.1$): the modulation of the potential is enlarged, however the overall tilt has been inverted to the left. Parameters ($\Delta\omega=0.05; Q=-1$).

sible for the constant growth of the phase difference $\phi$ between the uncoupled oscillators. With the onset of coupling the form of the potential (29) is modified in two different ways. First, the interaction leads to a periodic modulation of the potential, $\epsilon(2\cos\phi - Q\sin\phi)$, which is closely related to the possibility of synchronization since it tends to lock the phase difference into one of the arising local minima of $V(\phi)$. On the other hand, if $Q\neq 0$ an increase of coupling also gives rise to a change in the overall tilt of the potential, $\Delta\omega+\epsilon Q$ (see Fig. 9). It is this secondary effect of coupling, i.e. the tilting of the potential, which is responsible for the emergence of anomalous effects. The mean slope of the potential depends on a balance between the natural frequency mismatch $\Delta\omega$ and the term $\epsilon Q$. Without loss of generality assume that $\Delta\omega>0$ so that for $\epsilon=0$ the tilt is to the right. Then, if $Q$ is positive, with increase of coupling the overall tilt of the potential to the right is even enlarged. This leads to anomalous desynchronization. On the other hand, if $Q$ is negative then with increasing coupling strength the overall tilt is reduced, which corresponds to anomalous synchronization enhancement. For the specific coupling strength $\epsilon_s=-\Delta\omega/Q$ the overall tilt becomes zero, so that $\Delta\Omega(\epsilon_s)=0$ (corresponding to the shift of the locking plateau $\omega_s$ in Fig. 7b). If the coupling strength is increased further, $\epsilon>\epsilon_s$ then the overall tilt is inverted.

This inversion of the average slope of the potential is the origin of the phenomenon of natural frequency inversion. However, frequency inversion cannot arise in the deterministic system (29) because with increasing $\epsilon$ necessarily also the periodic modulation is switched on. This is expressed by the relation $\epsilon_c<\epsilon_s$, which can easily be proven from the synchronization threshold Eq. (25). This means that the synchronization threshold $\epsilon_c$ of the noiseless system is always smaller than the special coupling level $\epsilon_s$ where the tilt disappears. Therefore, even though the overall tilt of the potential eventually becomes negative, the phase difference $\phi$ first becomes trapped in one of the local potential minima. Only under stochastic influence the phase difference is able to jump over the potential barrier and so that the negative tilt of the potential can become effective.

## VI. CONCLUSION

In this paper we have investigated the synchronization between two nonisochronous oscillators. As we have shown, compared to isochronous systems such oscillators show marked differences in their response to the onset of coupling. Most notably, nonisochronous oscillators are linearly detuned in the presence of small coupling. Similar detuning has often been reported in natural systems. For example, it is known already for a long time that in real living oscillators, such as in the mammalian intestine and heart, the cells forming a tissue oscillate with frequencies that are different and usually are larger than the frequencies of the uncoupled, isolated cells [1,30]. If two nonisochronous oscillators are coupled this linear detuning is superimposed upon the usual coupling-induced attraction of the rotation frequencies. As a consequence, the synchronization of two oscillators with non-vanishing nonisochronicites can be intricate and "anomalous effects" may arise where coupling is able to increase, decrease, or even invert the natural frequency difference. Anomalous synchronization has been observed in a large class of limit cycle and chaotic oscillators (e.g., predator-prey systems, Van-der-Pol oscillator, Rössler system, Landau-Stuart systems, Chua-circuits etc.).

In the simplest case, the amount of anomalous synchronization depends on the difference in the nonisochronicity of the two oscillators. However, even though many natural oscillators are characterized by nonvanishing values of nonisochronicity, the differences usually are rather small. Our numerical studies in a large class of different systems indicate that this is a general rule. This may be the reason why so far anomalous effects have not been recognized very often. Here, we have demonstrated that the situation very different in the presence of asymmetrical couplings, where anomalous effects are considerably enlarged. Many or most studies of coupled oscillators are assuming only symmetrical interactions. However, in any realistic system symmetric coupling must be considered to be the exception. Coupling asymmetry can arise for a large number of different reasons. One common example is a heterogeneity in the size or some other extensive attribute of the oscillators, such as mass, capacity or volume. In this case, even if the interaction is microscopically symmetric, the nonidentical oscillators effectively may experience asymmetric forces [34]. Take for example a system of two coupled predator-prey systems. Assume that the interaction between the two population patches is through diffusive dispersal of individuals, which means that it is simply determined by the density difference between the two patches. In practice, the two patches will have different size. Therefore, one migrating individual will make a different contribution to the density of its new patch after migration. As a consequence, even though the microscopic process of diffusion is symmetric, the effective coupling to describe the population densities depends on the two patch sizes and in general will be asymmetric. Similar arguments apply for a large class of other systems. Take, for example, a pair of coupled pendulums with different masses. Even though the basic interaction for the exchange of momentum follows Newton's law of "actio=reactio," the coupling function in terms of velocities (and thus also frequencies) depends on the mass ratio between the two pendulums.

A second focus of this study is the constructive interplay between anomalous synchronization and noise. As we have shown, with the introduction of noise anomalous effects usually are strongly amplified. The reason is that anomalous effects increase with the coupling strength. However, without noise for large coupling the oscillators are synchronized before anomalous effects can become effective. This is different with the inclusion of noise because then the dynamics of the interacting oscillators are not fully correlated even for large coupling strength. This allows for strong anomalous effects and can give rise to many unusual results. For example, the frequency locking between two noisy oscillators may set in for oscillators with a non-zero frequency mismatch. Further, we have shown that under the influence of noise coupling is able to desynchronize two identical oscillators.

Taking into account that all major ingredients of this investigation, namely nonisochronicity, coupling asymmetry and noise, are ubiquitous in natural systems, we argue that the presented results should be of relevance for many applications. For example, anomalous synchronization allows for the possibility of synchronization control. With a careful choice of oscillator parameters or coupling direction anomalous effects can be used to either enhance or inhibit the synchronization between the oscillators. Therefore the effect is of potential use for engineering applications, but should also play a prominent role in living systems, where evolution may have selected parameter sets in such a way as to support biologically advantageous synchronization properties.

**ACKNOWLEDGMENTS**

This work was supported by the German Volkswagen-Stiftung. I thank E. Montbrio for helpful discussions.

## APPENDIX: NONISOCHRONICITY AND SHEAR OF PHASE FLOW

Here, we briefly comment on the origin for the strikingly different values of $q_i$ in our model systems. As already mentioned before, the nonisochronicity $q$ of an oscillator is a measure for the change of its rotation frequency after a small perturbation. Suppose the existence of a coordinate system so that the state of the limit cycle system can be described in terms of its phase $\theta(t)$ and amplitude $r(t)$. Let $\omega_0$ denote the natural frequency at the unperturbed amplitude $r_0$. Without loss of generality we set $r_0=1$. If the oscillator is disturbed to rotate with the new amplitude $r\neq 1$ then close to its limit cycle the instantaneous frequency can be written as $\omega(r)\approx\omega_0+q(1-r^2)$ [2,19,20]. Here, the nonisochronicity $q$ is a measure for the shear of the phase flow close to the limit cycle. Thus, the presence of nonisochronicity only becomes important when the oscillator is perturbed off its limit cycle either through the effects of noise, or as in our case, through the interaction with other systems.

Assume now a system of two weakly coupled oscillators (1). It is well known that the net effect of a very weak interaction, $\epsilon\ll\epsilon_c$, results in an effecting damping of the dynamics. Therefore, in the interacting systems the amplitudes on average, are reduced to the smaller radius $r^2(\epsilon)=1-\epsilon$ [19,20]. If the phase flow close to the limit cycle has a shear then this effective reduction in amplitude has the consequence that with increasing $\epsilon$ the coupling modified frequency is detuned as $\Omega(\epsilon)\approx\omega(r(\epsilon))\approx\omega_0+\epsilon q$, which corresponds exactly to our previous result (16).

Most natural oscillators are characterized by positive values of $q$, which means that the oscillation frequency decreases with the amplitude $r$. Assume, for simplicity, a system in which the absolute value of the velocity $v$ of all trajectories in phase space is constant. In such an oscillating system $\omega(r)=v/r$ with $v=\omega_0 r_0$. Therefore, close to a supposed limit cycle of radius $r_0=1$ we have $\omega(r)\approx\omega_0+\omega_0(1-r)$. Here, we have a positive shear because a rotation with larger amplitude, $r+dr$, requires the trajectories to follow a larger distance in phase space, which implies an increase of the average rotation time, $dT=r/Tdr$.

The unusual large levels of nonisochronicity $q_i\approx 4\ldots 5$, which are exhibited in the limit-cycle system (5), imply that with an increase of amplitude the oscillations are drastically slowed down. In the predator-prey system this is indeed the case. To understand this we first note that the densities $x$ and $y$ must always be positive. Therefore, larger oscillation amplitudes can only be expressed at the maxima of the cycle, whereas during the minima the trajectories must be squeezed onto the 0-axis. Thus, for large amplitudes the predator-prey oscillations are nonuniform and in the phase plane are characterized by trajectories that closely approach the $x$ and $y$ axis. During these times of very small predatoror prey abundance, the oscillators are slowed down, since the velocity $v$ is very small close to zero and very fast for large densities. Therefore, if the amplitude of oscillation is reduced by the interaction with other systems then the minimal density levels become elevated and the bottleneck of slow velocities close to zero is less severe. As a consequence, in the predator-prey system the period can strongly increase with the oscillation amplitude, which again implies a large value of $q$.

In contrast, in the chaotic three trophic predator-prey model, we observe large negative values of nonisochronicity. These negative values of $q$ imply that rotations with larger amplitudes are accelerated. This can be explained due to the chaotic folding in the three dimensional phase plane of this model [14]. Recall, that the uniform rotation in this model basically takes plane in the two lower trophic layers, i.e. in the $(x,y)$-plane. Large amplitudes in the $(x,y)$-plane trigger a strong but fast excursion in the $z$-variable, which resets the trajectory to a new small amplitude in the $(x,y)$-plane (in this way generating the chaotic folding of the phase flow). Therefore, a large amplitude in the $(x,y)$-plane gives rise to a fast oscillation, which finally results in the negative value of $q_i$.

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